Etale Cohomology Talk #1 : Étale morphisms

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1: Intro

In this talk I will give various definitions of étale morphisms in various generalities and provide examples.

2: Nonsingular varieties

We will first give the definition of étale morphisms for nonsingular varieties over an algebraically closed field k.

Definition : Let W and V be nonsingular varieties over k, and let $f : W \to V$ be a regular map. f is said to be étale at $Q \in W$ if the indued map on tangent spaces $T_Q(W) \to T_{f(Q)}(V)$ is an isomorphism, and f is étale if it is étale at every point of W.

This proposition works well for this case.

Proposition : Let V be a nonsingular variety over k with coordinate ring A, and let W be a subvariety of $V \times \mathbb{A}^n$ defined by n equations $g_1, ..., g_n \in A[y_1, ..., y_n]$. The projection map $W \to V$ is étale at a point $(P, b_1, ..., b_n)$ if and only if the Jacobian $\left(\frac{\partial g_i}{\partial y_j}\right)_{i,j}$ is nonsingular at $(P, b_1, ..., b_n)$.

This has the following easy corollary.

Corollary : Consider a regular map $\varphi : \mathbb{A}^m \to \mathbb{A}^m$. Then φ is étale at $(a_1, ..., a_n)$ if and only if the Jacobian matrix

$$\frac{\partial X_i \circ \varphi}{\partial Y_j}(a_1, ..., a_m)$$

is invertible. Here X_i, Y_j refer to coordinate functions $\mathbb{A}^m \to \mathbb{A}^1$.

Remark : Note the similarity to the condition of inverse function theorem on manifolds.

Example use of Corollary : Consider $f : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto x^n$, so $\frac{df}{dx}(x) = nx^{n-1}$. So if *char k* divides n, then this map is nowhere étale. Otherwise, for $n \geq 2$ it is étale at all points other than 0.

Let's think about the Proposition a bit more when $n = 1$. In this case we have some nonsingular variety V with coordinate ring A, and consider a polynomial $f(t) = a_0 t^m + ... + a_m$ with coefficients in A, which we can think of as a continuous family of polynomials parametrized by points in V. Let W denote the subvariety of $V \times \mathbb{A}^1$ defined by $f(t) = 0$. The projection map $\pi : W \to V$ corresponds to the ring map $A \to A[t]/(f(t))$. The fiber over $P_0 \in V$ is the set of roots of

$$f(P_0; t) = a_0(P_0)t^m + ... + a_m(P_0) \in k[t]$$

Then can see that :

1. The fiber $\pi^{-1}(P_0)$ is finite if and only if P_0 is not a common zero of the functions a_i. So π is quasi-finite if and only if the ideal generated by $a_0, ..., a_m$ is A.

2. The map π is finite if and only if a_0 is a unit in A (makes $A[t]/f(t)$ a f.g. module over A)

3. The map π is étale at $(P_0; c)$ if and only if c is a simple root of $f(P_0; t)$ (being multiple root makes the derivative $\frac{df(P_0;t)}{dt}$ at $(P_0, c) = 0$)

These observations will motivate us to define étale morphisms for general schemes.

3: General affine varieties

We continue to work over an algebraically closed field k.

At singular points the tangent spaces are not well-behaved. So instead, we define étale morphisms using the notion of a tangent cone.

Definition 1 : Let V be the variety with coodinate ring $k[x_1, ..., x_n]/\mathfrak{a}$. Suppose that the origin P is contained in V. Let $\mathfrak{a}_*$ be the ideal generated by the lowest degree homogeneous parts of elements of $\mathfrak{a}$. Then we define the tangent cone at P, denoted $C_P(V)$, to be either $k[x_1, ..., x_n]/\mathfrak{a}_*$ or the associated spctrum Spec $k[x_1, ..., x_n]/\mathfrak{a}_*$. These are used interchangeably.

There is another way to define the tangent cone without having to translate points in the domain, and works for general schemes.

Definition 2 : Let $P \in V$ (variety or scheme) and consider the local ring $\mathcal{O}_P$ with maximal ideal $\mathfrak{m}$. Then we can take the tangent cone to be the associated graded ring

$$gr(\mathcal{O}_P) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$$

Remark : For varieties over an algebraically closed field, the two definitions coincide.

Definition : Let $f : W \to V$ be a regular map of varieties. f is said to be étale at $Q \in W$ if the induced map on the tangent cone $C_{f(Q)}(V) \to C_Q(W)$ is an isomorphism (as k-algebras).

Example : Let $W = \mathbb{A}^1, V = Y^2 - X^3 + X^2$, and $f : W \to V$ defined by $t \mapsto (t^2 - 1, t(t^2 - 1))$. The corresponding map of k-algebras is

$$k[x, y]/(y^2 - x^3 - x^2) \to k[t], x \mapsto t^2 - 1, y \mapsto t(t^2 - 1)$$

Let $Q \in W$ be 1, so $f(Q) = P \in V$ is 0. Then $\mathcal{O}_Q = k[t]_{(t-1)}$, with maximal ideal generated by $s = t - 1$, and this element generates $\mathfrak{m}/\mathfrak{m}^2$. There are no relations so we can easily see that $gr(\mathcal{O}_Q) = k[s]$. Next $\mathcal{O}_P = (k[x, y]/(y^2 - x^3 - x^2))_{(x,y)}$ with maximal ideal generated by x, y, and this element generates $\mathfrak{m}/\mathfrak{m}^2$. The only relation between them is in $\mathfrak{m}^2/\mathfrak{m}^3$, we have $0 = y^2 - x^2 - x^3 = y^2 - x^2$. Hence $gr(\mathcal{O}_P) = k[x, y]/(y^2 - x^2)$ (here we can see why the two definitions coincide). Then we can already see that the two k-algebras cannot be isomorphic since the 1st degree pieces of the two k-algebras have different dimensions.

Just for computation, the induced map $gr(\mathcal{O}_P) \to gr(\mathcal{O}_Q)$ sends

$$x \mapsto (s+1)^2 - 1 = s^2 + 2s = 2s \bmod \mathfrak{m}^2, y \mapsto (s+1)((s+1)^2 - 1) = (s+1)(s^2 + 2s) = s^3 + 3s^2 + 2s = 2s \bmod \mathfrak{m}^2$$

□

Prop : For a local homomorphism $A \to B$ of local rings the induced map on the associated graded rings is an isomorphism if and only if the induced map on completions is an isomorphism.

So using this proposition, we can alternately say that f is étale at Q if and only if the induced map on completions of the local rings $\hat{\mathcal{O}}_{f(Q)} \to \hat{\mathcal{O}}_Q$ is an isomorphism.

Definition : For general varieties over an arbitrary field, we say that a regular map $f : W \to V$ is étale at Q if the induced map $f_{\bar{k}} : W_{\bar{k}} \to V_{\bar{k}}$ is étale at the points of $W_{\bar{k}}$ mapping to Q.

4: Schemes

The definition for étale morphisms between schemes use the concept of flat and unramified :

Definition : Recall that given a ring map $A \to B$, B is a flat A-algebra if $\otimes_A B$ preserves injections (left exact). A morphism $f : Y \to X$ of schemes is flat if the induced maps $\mathcal{O}_{f(y)} \to \mathcal{O}_y$ of local rings are flat maps for all $y \in Y$.

Basically, this means that the dimension of the fibers are not changing.

Definition : Recall that given a local ring map $A \to B$, it is unramified if the image of $\mathfrak{m}_A$ generates $\mathfrak{m}_B$, and $B/\mathfrak{m}_B$ is a finite separable extension of $A/\mathfrak{m}_A$. A morphism $f : Y \to X$ of schemes is unramified if it is of finite type and if the maps $\mathcal{O}_{f(y)} \to \mathcal{O}_y$ of local rings are unramified for all y.

Remark : A finite type morphism $f : Y \to X$ is unramified if and only if $\Omega^1_{Y/X} = 0$.

Definition : A morphism $f : Y \to X$ of schemes is étale if it is flat and unramified.

Remark : Flat and unramified can be just checked on closed points of Y, so étaleness can be just checked on closed points.

Remark : By definition of finite type, we can just look at f on affine patches Spec $B \to$ Spec A. This is étale if the corresponding ring map $A \to B$ satisfies :

1. B is a f.g. A-algebra
2. B is a flat A-algebra
3. For all maximal ideals $\mathfrak{n}$ of B, $B_\mathfrak{n}/f(\mathfrak{p})B_\mathfrak{n}$ is a finite separable field extension of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, where $\mathfrak{p} = f^{-1}(\mathfrak{n})$

5: Useful Properties

Claim : Composition of étale morphisms is étale.

Claim : A base change of étale morphisms is étale.

Claim : If $f \circ g$ and f are étale, then g is étale. This is useful for talking about over categories.

Claim: For all $y \in Y, \mathcal{O}_y$ and $\mathcal{O}_x$ have the same Krull dimension.

Claim : An étale morphism is an open map. So an open immersion is étale, but a closed immersion is not étale unless it's mapping isomorphically onto a connected component of the codomain.

Claim : An étale morphism is quasi-finite.

Claim : An étale morphism $f : X \to Y$, if X is reduced/normal/regular, then so is Y respectively.

6: Examples :

Example : Suppose that $X \to$ Spec k us an étale morphism. Then $X = \amalg_i$Spec L_i, where each L_i is a finite, separable extension of Spec k.

Proof : Since étale implies quasi finite, X must be a disjoint union of finitely many points, i.e. a collection of some Spec L where L is a field extension of k. Condition 3 from the previous page say that the extension must be finite and separable. □

Example : Consider the map Spec $\mathbb{Z}_{(p)}[i] \to$ Spec $\mathbb{Z}_{(p)}$ where p is some prime. This is étale only if p is odd.

Proof : Since $\mathbb{Z}_{(p)}$ is a Dedekind domain, the inclusion $\mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)}[i]$ shows that this is a flat $\mathbb{Z}_{(p)}$-algebra, which takes care of condition 2. It is also generated by i, which takes care of Condition 1. Finally, the only max ideal of $\mathbb{Z}_{(p)}[i]$ is generated by p denoted as (p). This has preimage (p) in $\mathbb{Z}_{(p)}$. The field extension we care is $\mathscr{F}_p[i]/\mathscr{F}$. This is clearly finite, but not separable if $p = 2$.